Planar Critical Percolation

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4 Expected number of clusters intersecting a line segment

This chapter is based on [12] with Rob van den Berg.

We study critical percolation on a regular planar lattice. Let $E_G(n)$ be the expected number of open clusters intersecting or hitting the line segment $[0, n]$. (For the subscript G we either take $\mathbb{H}$, when we restrict to the upper halfplane, or $\mathbb{C}$, when we consider the full lattice).

Cardy [25] (see also Yu, Saleur and Haas [84]) derived heuristically that $E_{\mathbb{H}}(n) = An + \frac{\sqrt{3}}{4\pi}\log(n) + o(\log(n))$, where A is some constant. Recently Kovács, Iglói and Cardy derived in [59] heuristically (as a special case of a more general formula) that a similar result holds for $E_{\mathbb{C}}(n)$ with the constant $\frac{\sqrt{3}}{4\pi}$ replaced by $\frac{5\sqrt{3}}{32\pi}$.

In this chapter we give, for site percolation on the triangular lattice, a rigorous proof for the formula of $E_{\mathbb{H}}(n)$ above, and a rigorous upper bound for the prefactor of the logarithm in the formula of $E_{\mathbb{C}}(n)$.

4.1 Introduction

Consider critical bond percolation on $\mathbb{Z}^2$. Kovács, Iglói and Cardy [59] studied the expected number of clusters which intersect the boundary of a polygon. The leading order is the size n of the boundary. The prefactor of this term is lattice dependent. Their main interest is in the first correction term (of order $\log n$). Their motivation came from relations with entanglement entropy in a diluted quantum Ising model. Using indirect and non-rigorous methods from conformal field theory and the q-state Potts model (letting $q \to 1$), they derived a (universal) formula for the prefactor of the logarithmic term.

A special case of their result is that of a line segment (treated in Section F of their paper). In their setup the line segment was placed in the full plane and they claim that the prefactor is equal to $\frac{5\sqrt{3}}{32\pi}$. Furthermore they refer to an earlier obtained result by Cardy in [25] (see also Yu, Saleur and Haas [84]) where the line segment was placed on the boundary of the half-plane. In the latter case the claim is that the prefactor equals $\frac{\sqrt{3}}{4\pi}$. Also this latter result was obtained by non-rigorous arguments using q-state Potts models.

This motivated us to try to find rigorous and more direct proofs of these results (starting with the case of line segments). Since the prefactors are believed to be universal it is natural to consider the most well studied percolation model, site percolation on the triangular lattice.

Because conformal invariance plays a role, it is convenient to identify the plane with the set $\mathbb{C}$ of complex numbers. We embed the triangular lattice $\mathbb{T}$ in the half-plane $\mathbb{H} = \{z : \Im z \geq 0\}$ or the full plane $\mathbb{C}$ with vertex set $\{m + n\mathbf{j} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ (resp. $\{m + n\mathbf{j} : m, n \in \mathbb{Z}\}$), where $\mathbf{j} = e^{\frac{\pi}{3}\mathbf{i}}$. We denote the probability measure by $\mathbb{P}_{\mathbb{H}}$ (resp. $\mathbb{P}_{\mathbb{C}}$) and the expectation by $\mathbb{E}_{\mathbb{H}}$ (resp. $\mathbb{E}_{\mathbb{C}}$). For subsets $A, B \subset \mathbb{C}$ we denote by $A \leftrightarrow B$ the event that there are vertices x, y on the triangular lattice, with $x \in A, y \in B$, which are connected by a path of open vertices. with some abuse of notation we denote, for any $x \in \mathbb{C}$, the set $\{x\}$ by x. A cluster is a collection connected vertices. Consider the line segment $[1, n]$ on $\mathbb{R}$, containing n vertices. We are interested in

$$E_G(n) := \mathbb{E}_G[\,|\{C \in \mathcal{C}_G : C \cap [1, n] \neq \emptyset\}|\,],$$

where $\mathcal{C}_G$ is the collection of all clusters in the triangular lattice on the lattice $G = \mathbb{H}, \mathbb{C}$.

It is easy to derive the leading (of order n) term: see the Remark a few paragraphs below Theorem 4.1.1. In the case of the half-plane we could obtain a rigorous proof for the earlier mentioned logarithmic correction term. In the case of the full plane we only obtained a logarithmic upper bound for the correction term. (We do not see a method how to prove the precise prefactor $\frac{5\sqrt{3}}{32\pi}$ given in [59]; even finding a non-trivial lower bound is, in our opinion, a challenging problem).

More precisely, our main contribution is a rigorous proof of the following:

Theorem 4.1.1.

$$\textit{(a)} \quad E_{\mathbb{H}}(n) = n \cdot (\mathbb{P}_{\mathbb{H}}(1 \not\leftrightarrow (-\infty, 0]) - \frac{1}{2}) + \frac{\sqrt{3}}{4\pi} \log(n) + o(\log(n))$$

and

$$\textit{(b)} \qquad \limsup_{n \to \infty} \frac{E_{\mathbb{C}}(n) - n \cdot (\mathbb{P}_{\mathbb{C}}(1 \not\leftrightarrow (-\infty, 0]) - \frac{1}{2})}{\log(n)} \leq \frac{8}{5} \cdot \frac{\sqrt{3}}{4\pi}.$$

We now describe the first steps of the strategy to derive the result above. This will also give some insight, where the log comes from. First rewrite the number of clusters as follows

$$\begin{aligned} |\{C \in \mathcal{C}_G : C \cap [1, n] \neq \emptyset\}| &= \mathbf{1}\{1 \text{ open}\} + \sum_{k=2}^{n} \mathbf{1}\{k \not\leftrightarrow [1, k-1], k \text{ open}\} \\ &= 1 + \sum_{k=2}^{n} \mathbf{1}\{k \not\leftrightarrow [1, k-1]\} - \sum_{k=1}^{n} \mathbf{1}\{k \text{ closed}\} \end{aligned}$$

So

$$
\begin{aligned}
E_G(n) & \\
= \quad & 1-\frac{1}{2}n+\sum_{k=2}^{n}\big(\mathbb{P}_G(k \not\leftrightarrow (-\infty,k-1])+\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\})\big) \\
= \quad & 1-\frac{1}{2}n+(n-1)\cdot(\mathbb{P}_G(1 \not\leftrightarrow (-\infty,0])) \\
& +\sum_{k=2}^{n}\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\}).
\end{aligned}
$$

Remark: Since $\mathbb{P}_G((-\infty,0] \leftrightarrow [k,\infty)) \to 0$ as $k \to \infty$, this implies that the leading term of $E_G(n)$ is $n(\mathbb{P}_G(1 \not\leftrightarrow (-\infty,0]) - \frac{1}{2})$.

Let us introduce the following notation:

$$L_G(n) := \frac{1}{\log(n)}\sum_{k=2}^{n}\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\}).$$

That is,

$$L_G(n) = \frac{E_G(n)-1+\frac{1}{2}n-(n-1)\cdot(\mathbb{P}_G(1 \not\leftrightarrow (-\infty,0]))}{\log(n)}.$$

Hence Theorem 4.1.1 is equivalent to
(a) $\lim_{n\to\infty} L_{\mathbb{H}}(n) = \frac{\sqrt{3}}{4\pi}$ and
(b) $\limsup_{n\to\infty} L_{\mathbb{C}}(n) \leq \frac{8}{5}\cdot\frac{\sqrt{3}}{4\pi}$.

Take $\varepsilon > 0$. We will introduce $M = M(n,\varepsilon) \in \mathbb{N}$ and a sequence $a(i) = a(i,n,\varepsilon)$ for $1 \leq i \leq M+1$, such that

$$a(M+1) = n.$$

With these values we split up the sum in $L_G(n)$ in the following terms. For all $1 \leq i \leq M$,

$$f_i := \sum_{k=a(i)+1}^{a(i+1)}\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\}). \tag{4.1}$$

and

$$f_0 := \sum_{k=2}^{a(1)}\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\}). \tag{4.2}$$

Then

$$L_G(n) = \frac{f_0}{\log(n)} + \frac{1}{\log(n)}\sum_{i=1}^{M} f_i.$$

The idea is now, roughly speaking, to choose $a(i,n,\varepsilon)$ so that the ratio of two consecutive ones equals $1+\varepsilon$ and choose M such that $a(1,n,\varepsilon)$ goes to infinity as $n \to \infty$, but is of a smaller order than $\log(n)$.

Than obviously the term $f_0/\log(n)$ is negligible. We will see that M is more or less of the order $\log(n)/\varepsilon$. The existence of the limit $\lim_{n\to\infty} L_G(n)$ would follow if we can show that, for ε close to zero, f_i is approximately a constant times ε as $n \to \infty$.

In the case that $G = \mathbb{H}$, we will see in Section 4.3.1 that this strategy indeed leads to the existence, and even the value, of the limit of $L_{\mathbb{H}}(n)$ as $n \to \infty$. Unfortunately in the full-plane it only leads to the upper bound stated in Theorem 4.1.1 (b), as we will see in Section 4.3.2.

Now we make the above choices precise. We define

$$M := \left\lfloor \frac{\log(n) - \frac{1}{2}\log(\log(n))}{\log(1+\varepsilon)} \right\rfloor \tag{4.3}$$

and for $i \in \{-1, \cdots, M-1\}$

$$a(M-i, n, \varepsilon) := \left\lfloor \frac{n}{(1+\varepsilon)^{i+1}} \right\rfloor \tag{4.4}$$

or alternatively, for $j \in \{1, \cdots, M+1\}$

$$a(j, n, \varepsilon) := \left\lfloor \frac{n}{(1+\varepsilon)^{M-j+1}} \right\rfloor .$$

Note that than $a(1, n, \varepsilon)$ is of order $\sqrt{\log(n)}$. To examine f_i it is useful to rewrite it in terms of an expectation as follows. Let

$$T(i) := \sum_{k=a(i)+1}^{a(i+1)} \mathbf{1}\{k \not\leftrightarrow [1, k-1] \text{ and } k \leftrightarrow (-\infty, 0]\}. \tag{4.5}$$

Then $f(i) = \mathbb{E}_G[T(i)]$. Hence

$$L_G(n) = \frac{f_0}{\log(n)} + \frac{1}{\log(n)} \sum_{i=1}^{M} \mathbb{E}_G[T(i)]. \tag{4.6}$$

4.2 Preliminaries

In this section we state some results, which we will use in the proof of our main result, Theorem 4.1.1. First some additional notation. We use the following notation for the probabilities of so-called arm-events. Let, for $m < n \in \mathbb{N}$

$$\pi_1(m, n) := \mathbb{P}_{\mathbb{H}}([-m, m]^2 \leftrightarrow \mathbb{H} \setminus [-n, n]^2) \tag{4.7}$$

and let $\pi_3(m, n)$ be the probability of having two disjoint closed paths, and an open path, from $[-m, m]^2$ to $\mathbb{H} \setminus [-n, n]^2$. The following lemma is well known (see for example Theorems 21 and 22 in [65]).

Lemma 4.2.1. *There exist constants $C_1, C_2 > 0$ and $\alpha \leq 1/2$ such that, for all $m < n$*

$$\pi_1(m, n) \leq C_1 \left(\frac{m}{n}\right)^{\alpha}, \qquad \pi_3(m, n) \leq C_2 \left(\frac{m}{n}\right)^{2}.$$

In fact, much more precise results for these probabilities are known, but will not be used in this chapter.

In the rest of this section, for a domain $D \subsetneq \mathbb{C}$ and $n \in \mathbb{N}$ the notation nD denotes the set $\{n \cdot u : u \in D\}$. For points a_1, a_2 on the boundary of D we denote by $[a_1, a_2]$ the part of the boundary of D between a_1 and a_2 in the counter clockwise direction. Furthermore we generalize the notation slightly, namely by $\mathbb{P}_D$ (and $\mathbb{E}_D$) we will denote the probability measure for percolation restricted to the triangular lattice on D. In this setting two intervals $[a_1, a_2]$ and $[a_3, a_4]$ on the boundary are said to be connected if the there are vertices x, y on the lattice inside D, which are connected, and are such that x has an edge which crosses $[a_1, a_2]$ and y has an edge which crosses $[a_3, a_4]$.

The first theorem is the famous Cardy's formula, which was proved by Smirnov in [77].

Theorem 4.2.2 (Cardy's formula, [77]). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and $\phi : D \to \mathbb{H}$ a conformal map. Let a_1, a_2, a_3, a_4 be ordered points on the boundary of D. We have*

$$\lim_{n\to\infty} \mathbb{P}_{nD}([na_1, na_2] \leftrightarrow [na_3, na_4]) = \frac{2\pi\sqrt{3}}{\Gamma\left(\frac{1}{3}\right)^3}\lambda^{1/3} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \lambda\right),$$

where λ is the cross-ratio

$$\lambda = \frac{(\phi(a_1) - \phi(a_2))(\phi(a_4) - \phi(a_3))}{(\phi(a_1) - \phi(a_3))(\phi(a_4) - \phi(a_2))}. \tag{4.8}$$

This theorem concerns crossing probabilities of generalized rectangles in one 'direction'. The following theorem gives a formula for probabilities of crossings in two directions. It is called after Watts, who proposed the formula in 1996. The first rigorous proof was by Dubédat [34]. An alternative proof was obtained by Schramm (see [73]).

Theorem 4.2.3 (Watts' formula, [34, 73]). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and $\phi : D \to \mathbb{H}$ a conformal map. Let a_1, a_2, a_3, a_4 be ordered points on the boundary of D. We have*

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}_{nD}([na_1, na_2] \leftrightarrow [na_3, na_4] \text{ and } [na_4, na_1] \leftrightarrow [na_2, na_3])\\ &= \frac{2\pi\sqrt{3}}{\Gamma\left(\frac{1}{3}\right)^3}\lambda^{1/3} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \lambda\right) - \frac{\sqrt{3}}{2\pi}\lambda \cdot {}_3F_2\left(1, 1, \frac{4}{3}; \frac{5}{3}, 2; \lambda\right).\end{aligned}$$

where λ is the cross-ratio (4.8).

The last theorem we state here concerns the expected number of crossing clusters of a rectangle. It was predicted by Cardy [25] and by Simmons, Kleban and Ziff [75]. A proof was given by Smirnov and Hongler in [49]. Here $N(nD, a_1, a_2, a_3, a_4)$ denote the number of clusters in nD which connect $[na_1, na_2]$ with $[na_3, na_4]$.

Theorem 4.2.4 ([49]). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and $\phi : D \to \mathbb{H}$ a conformal map. Let a_1, a_2, a_3, a_4 be ordered points on the boundary of D. We have*

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_{nD}[N(nD, a_1, a_2, a_3, a_4)] \\
&= \frac{2\pi\sqrt{3}}{\Gamma\left(\frac{1}{3}\right)^3}\lambda^{1/3} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \lambda\right) - \frac{\sqrt{3}}{4\pi}\lambda \cdot {}_3F_2\left(1, 1, \frac{4}{3}; \frac{5}{3}, 2; \lambda\right) + \frac{\sqrt{3}}{4\pi}\log\left(\frac{1}{1-\lambda}\right).
\end{aligned}$$

where λ is the cross-ratio (4.8).

4.3 Proof of Theorem 4.1.1

Recall from the introduction that Theorem 4.1.1 is equivalent to
(a) $\lim_{n\to\infty} L_{\mathbb{H}}(n) = \frac{\sqrt{3}}{4\pi}$ and
(b) $\limsup_{n\to\infty} L_{\mathbb{C}}(n) \le \frac{8}{5} \cdot \frac{\sqrt{3}}{4\pi}$.

Recall the definition (4.5) of $T(i)$. We begin this section with a lemma which says that, to prove the convergence of $L_G(n)$ as $n \to \infty$, it is sufficient to prove the convergence of $\varepsilon^{-1}\mathbb{E}_G[T(i)]$.

Lemma 4.3.1. *The following inequalities hold.*

$$\limsup_{n\to\infty} L_G(n) \le \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \max_{1\le i\le M} \frac{\mathbb{E}_G[T(i)]}{\varepsilon} \tag{4.9}$$

and

$$\liminf_{n\to\infty} L_G(n) \ge \liminf_{\varepsilon\to 0} \liminf_{n\to\infty} \min_{1\le i\le M} \frac{\mathbb{E}_G[T(i)]}{\varepsilon}. \tag{4.10}$$

Proof: Recall (4.6) and the definitions of $M, a(i), f_i$ in (4.1) - (4.4). To prove (4.9), first note that $0 \le f_0 \le a(1, n, \varepsilon)$ and M was chosen such that $a(1, n, \varepsilon) \approx \sqrt{\log(n)}$, hence

$$\lim_{n\to\infty} \frac{f_0}{\log(n)} = 0.$$

Thus it is enough to prove that

$$\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \left(\sum_{i=1}^{M} \frac{\mathbb{E}_G[T(i)]}{\log(n)}\right) \le \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \max_{1\le i\le M} \frac{\mathbb{E}_G[T(i)]}{\varepsilon}. \tag{4.11}$$

Hereto, note that it is also easy to see from the definition of M that, for fixed $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{M}{\log(n)} = \frac{1}{\log(1+\varepsilon)}.$$

For all $\varepsilon > 0$ we have

$$\begin{aligned}
\limsup_{n\to\infty} \sum_{i=1}^{M} \frac{\mathbb{E}_G[T(i)]}{\log(n)} &\le \limsup_{n\to\infty} \left(\frac{M}{\log(n)} \max_{i\le M} \mathbb{E}_G[T(i)]\right) \qquad (4.12) \\
&\le \frac{1}{\log(1+\varepsilon)} \cdot \varepsilon \cdot \limsup_{n\to\infty} \left(\max_{i\le M} \frac{\mathbb{E}_G[T(i)]}{\varepsilon}\right).
\end{aligned}$$

Next note that

$$\limsup_{\varepsilon\to 0}\left(\frac{\varepsilon}{\log(1+\varepsilon)}\cdot\limsup_{n\to\infty}\left(\max_{i\le M}\frac{\mathbb{E}_G[T(i)]}{\varepsilon}\right)\right)=\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\left(\max_{i\le M}\frac{\mathbb{E}_G[T(i)]}{\varepsilon}\right).$$

This together with (4.12) implies (4.11) and completes the proof of (4.9).

The inequality in (4.10) follows in a similar way and we omit it. □

4.3.1 Proof of Theorem 4.1.1 (a)

First note that it is easy to see that $\{T(i)\ge 1\}$ if and only if there is an open and a closed path from $(-\infty,0]$ to $[a(i)+1,a(i+1)]$ and the closed path is below the open path. Furthermore the event $\{T(i)\ge m\}$ is equal to the event that there are $2m$ alternating paths between the aforementioned intervals, starting, from below, with a closed path. Thus the BK inequality implies that

$$\mathbb{P}_{\mathbb{H}}(T(i)\ge m)\le(\mathbb{P}_{\mathbb{H}}(T(i)\ge 1))^m. \tag{4.13}$$

Hence

$$\begin{aligned}
\mathbb{E}_{\mathbb{H}}[T(i)] &= \sum_{m=1}^{\infty}\mathbb{P}_{\mathbb{H}}(T(i)\ge m) && (4.14)\\
&\le \mathbb{P}_{\mathbb{H}}(T(i)\ge 1)+\sum_{m=2}^{\infty}(\mathbb{P}_{\mathbb{H}}(T(i)\ge 1))^m\\
&= \mathbb{P}_{\mathbb{H}}(T(i)\ge 1)+\frac{(\mathbb{P}_{\mathbb{H}}(T(i)\ge 1))^2}{1-\mathbb{P}_{\mathbb{H}}(T(i)\ge 1)}.
\end{aligned}$$

It is well-known from standard RSW arguments that $\mathbb{P}_{\mathbb{H}}(T(i)\ge 1)$ goes, uniformly in i and n, to 0 as $\epsilon\to 0$. Hence the 'error term' (i.e. the second term in the r.h.s. of the equation array above) is negligible w.r.t. the main term (i.e. the first term in the r.h.s.). By this, Lemma 4.3.1, the fact that $a(1)\to\infty$ as $n\to\infty$, and the ratio between consecutive $a(i)$'s, it is sufficient to prove that

$$\lim_{k\to\infty}\mathbb{P}_{\mathbb{H}}(W_k)=\frac{\sqrt{3}}{4\pi}\varepsilon+o(\varepsilon), \tag{4.15}$$

where W_k denotes the event that there is an open and a closed path from $(-\infty,0]$ to $[k,k(1+\varepsilon)]$ and the closed path is below the open path.

Let W_k' be the event that there is an open and a closed path from $(-\infty,0]$ to $[k,k(1+\varepsilon)]$. (So, informally speaking, W_k' is the same as W_k without the condition on which path is above or below). Using that (by duality), there is either an open path from $[0,k]$ to $[k(1+\varepsilon),\infty)$ or a closed path from $(-\infty,0]$ to $[k,k(1+\varepsilon)]$, we have

$$\begin{aligned}
&\mathbb{P}_{\mathbb{H}}((-\infty,0]\leftrightarrow[k,k(1+\varepsilon)]\text{ and }[0,k]\leftrightarrow[k(1+\varepsilon),\infty)) && (4.16)\\
&\qquad=\mathbb{P}_{\mathbb{H}}((-\infty,0]\leftrightarrow[k,k(1+\varepsilon)])-\mathbb{P}_{\mathbb{H}}(W_k').
\end{aligned}$$

The limits as $k\to\infty$ of the first probability in the r.h.s. and the probability in the l.h.s. are obtained by Theorem 4.2.2 and Theorem 4.2.3 respectively, and we get

$$\begin{aligned}\lim_{k\to\infty}\mathbb{P}_{\mathbb{H}}(W'_k) &= \frac{\sqrt{3}}{2\pi}\cdot\frac{\varepsilon}{1+\varepsilon}\cdot{}_3F_2\left(1,1,\frac{4}{3};\frac{5}{3},2;\frac{\varepsilon}{1+\varepsilon}\right) \\ &= 2\frac{\sqrt{3}}{4\pi}\cdot\varepsilon+o(\varepsilon).\end{aligned} \tag{4.17}$$

Finally, let $\tilde{W}_k$ denote the event obtained from W_k by replacing 'open' by 'closed' and vice versa. Since W_k and $\tilde{W}_k$ have the same probability and $W'_k = \tilde{W}_k \cup W_k$, we have

$$\mathbb{P}_{\mathbb{H}}(W'_k) = 2\mathbb{P}_{\mathbb{H}}(W_k) - \mathbb{P}_{\mathbb{H}}(W_k \cap \tilde{W}_k).$$

Since $W_k \cap \tilde{W}_k$ is contained in the disjoint occurrence of W'_k and the event that there is an open or closed path from $(-\infty, 0]$ to $[k, k(1+\varepsilon)]$, its probability is negligible w.r.t. that of W'_k, and we get

$$\lim_{k\to\infty}\mathbb{P}_{\mathbb{H}}(W_k) = \frac{1}{2}\lim_{k\to\infty}\mathbb{P}_{\mathbb{H}}(W'_k),$$

which by (4.17) is equal to $\frac{\sqrt{3}}{4\pi}\cdot\varepsilon + o(\varepsilon)$. As we saw (see the argument above (4.15)) this proves Theorem 4.1.1 (a). □

4.3.2 Proof of Theorem 4.1.1 (b)

We will bound the relevant probabilities (concerning the full plane) by the probabilities of certain connection events in the half-plane. We do this by cutting along the real line from $-\infty$ up to $a(i+1)$. Let us make the cutting precise. Let

$$L(i) := (-\infty, a(i+1)],$$

we define the new lattice to be the triangular lattice on $\mathbb{C} \setminus L(i)$. This is the full triangular lattice, without the vertices (and their edges) on $L(i)$. Let us denote the corresponding probability measure, concerning percolation on this sublattice, by $\tilde{\mathbb{P}}_i$ (and expectation by $\tilde{\mathbb{E}}_i$). Let the boundary $\partial_{\mathbb{T}}[a,b]$ of an interval $[a,b] \subset L(i)$ be the vertices v of $\mathbb{T}$ which are not in the interval $[a,b]$ but have a neighbouring vertex which is on the interval $[a,b]$. Let $\tilde{T}(i)$ be the number of clusters which connect $\partial_{\mathbb{T}}[a(i)+1, a(i+1)]$ with $\partial_{\mathbb{T}}(-\infty, 0]$ but are not connected with $\partial_{\mathbb{T}}[1, a(i)]$.

With this definition of $\tilde{T}(i)$ 'almost all' the open connections counted in $T(i)$ are counted in $\tilde{T}(i)$ as well; however, there are exceptions. In these exceptional cases there is an open connection from $(-\infty, 0]$ to $[a(i)+1, a(i+1)]$ which is not connected to $[1, a(i)]$ on $\mathbb{T}$, but ís connected to $\partial_{\mathbb{T}}[1, a(i)]$ on $\mathbb{C} \setminus L(i) \cap \mathbb{T}$. See Figure 3. More precisely, we define

$$B(i) := \bigcup_{k\in[1,a(i)]\cap\mathbb{T}} \left(B_u(i,k) \cup B_l(i,k)\right),$$

where $B_u(i,k)$ is the event that, on $\mathbb{H} \cap \mathbb{T}$, there are closed paths from k to $(-\infty, 1]$ and from k to $[a(i), \infty)$ and open paths from one of the vertices $k + \mathbf{j}$ and $k - 1 + \mathbf{j}$

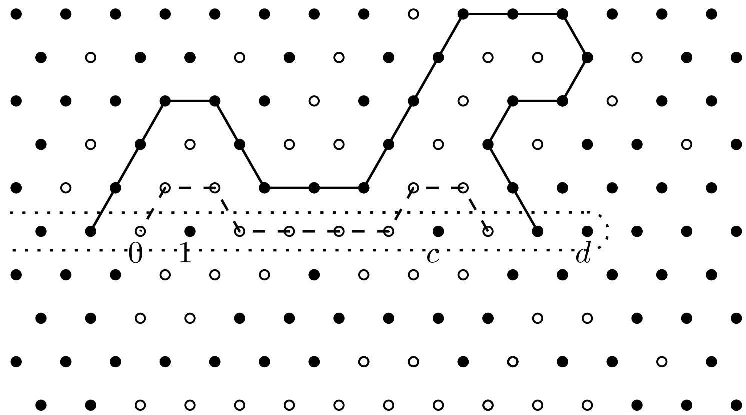

Figure 4.1: The event $B(i)$ occurs. Filled circles are open and empty circles are closed. $c = a(i), d = a(i+1)$.

to $(-\infty, 0]$ and to $[a(i)+1, \infty)$. (The open paths are not necessarily disjoint). The event $B_l(i,k)$ is defined similarly on the lower half-plane.

We have

$$\mathbb{E}_{\mathbb{C}}[T(i)] \leq \tilde{\mathbb{E}}_i[\tilde{T}(i)] + 2\mathbb{P}_{\mathbb{C}}(B(i)). \tag{4.18}$$

To bound $\mathbb{P}_{\mathbb{C}}(B(i))$ we use the first inequality of Lemma 4.2.1 for those k in the definition of $B(i)$ that are 'close to' 1 or $a(i)$, and the other inequality in that lemma for the other k's. More precisely, we fix a constant $\beta \in (0,1)$, and let $r(a(i)) := \lceil a(i)^\beta \rceil$. Then,

$$\begin{aligned} \mathbb{P}_{\mathbb{C}}(B(i)) &\leq 4\pi_1(r(a(i)), a(i)) + 4 \sum_{k=r(a(i))+1}^{\lceil \frac{1}{2}a(i) \rceil} \pi_3(1,k) \\ &\leq 4C_1 \left(\frac{r(a(i))}{a(i)} \right)^\alpha + 4 \sum_{k=r(a(i))+1}^{\infty} C_2 \left(\frac{1}{k} \right)^2, \end{aligned} \tag{4.19}$$

where the factor 4 comes from symmetry considerations. Hence, there exist constants $C_3, C_4 > 0$ such that

$$\mathbb{P}_{\mathbb{C}}(B(i)) \leq C_3 \left(a(i)^{\beta-1} \right)^\alpha + \frac{C_4}{a(i)^\beta} \tag{4.20}$$

Note that, since $a(1)$ (the smallest of the $a(i)$'s) tends to ∞ as $n \to \infty$, and $C_3(x^{\beta-1})^\alpha + \frac{C_4}{x^\beta}$ tends to 0 as $x \to \infty$, the contribution of $\mathbb{P}_{\mathbb{C}}(B(i))$ to the r.h.s. of (4.9) is 0.

Next we consider the term $\tilde{\mathbb{E}}_i[\tilde{T}(i)]$. Let $S(i)$ denote the number of closed clusters connecting $\partial_{\mathbb{T}}[a(i)+1, a(i+1)]$ with $\partial_{\mathbb{T}}(-\infty, 0]$. Observe that

$$\tilde{T}(i) = S(i) - \mathbf{1}\{S(i) \geq 1\}.$$

Thus it follows immediately, that

$$\tilde{\mathbb{E}}_i[\tilde{T}(i)] = \tilde{\mathbb{E}}_i[S(i)] - \tilde{\mathbb{P}}_i(S(i) \geq 1). \tag{4.21}$$

To complete the proof we will use Theorem 4.2.4. Therefore we consider the domain $\mathbb{C} \setminus L(i)$ and scale it by $a(i)$. (As noted before, $a(1)$ goes to ∞ as $n \to \infty$). This gives the conformal rectangle $\mathbb{C} \setminus (-\infty, 1+\varepsilon)$ with 'corners' 0^+, 0^-, 1^+ and 1^- (where, for $x < 1+\varepsilon$, x^+ and x^- denote the 'copy' of x in the upper and the lower half-plane respectively). To apply Theorem 4.2.4 we need the cross-ratio, which can be computed as follows: Consider the conformal map

$$\varphi(z) := \mathbf{i}\sqrt{z-1-\varepsilon}$$

which maps $\mathbb{C} \setminus (-\infty, 1+\varepsilon)$ onto the upper half-plane. The cross-ratio is

$$\lambda(\varepsilon) = \frac{(\varphi(1^+)-\varphi(1^-))(\varphi(0^-)-\varphi(0^+))}{(\varphi(0^+)-\varphi(1^-))(\varphi(0^-)-\varphi(1^+))}.$$

It is easy to see that

$$\varphi(0^-) = -\sqrt{1+\varepsilon}, \quad \varphi(1^-) = -\sqrt{\varepsilon}, \quad \varphi(1^+) = \sqrt{\varepsilon}, \quad \varphi(0^+) = \sqrt{1+\varepsilon}.$$

Hence

$$\begin{aligned} \lambda(\varepsilon)^2 &= \frac{16\varepsilon(1+\varepsilon)}{(\sqrt{1+\varepsilon}+\sqrt{\varepsilon})^4} \\ &= 16\varepsilon + o(\varepsilon). \end{aligned} \tag{4.22}$$

Applying Theorem 4.2.4 we conclude that, as $n \to \infty$, $\tilde{\mathbb{E}}_i[S(i)]$ converges (uniformly in the $i's$ with $1 \le i \le M(n)$), to

$$\begin{aligned} &\frac{2\pi\sqrt{3}}{\Gamma(\frac{1}{3})^3}\lambda(\varepsilon)^{1/3} \cdot {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};\lambda(\varepsilon)\right) \\ &- \frac{\sqrt{3}}{4\pi}\lambda(\varepsilon) \cdot {}_3F_2\left(1,1,\frac{4}{3};\frac{5}{3},2;\lambda(\varepsilon)\right) + \frac{\sqrt{3}}{4\pi}\log\left(\frac{1}{1-\lambda(\varepsilon)}\right). \end{aligned}$$

The first term is exactly the limit $\tilde{\mathbb{P}}_i(S(i) \ge 1)$ as $n \to \infty$ (Cardy's formula). Hence by noting that

$$-\frac{\sqrt{3}}{4\pi}\lambda \cdot {}_3F_2\left(1,1,\frac{4}{3};\frac{5}{3},2;\lambda\right) + \frac{\sqrt{3}}{4\pi}\log\left(\frac{1}{1-\lambda}\right) = \frac{\sqrt{3}}{4\pi}\cdot\frac{1}{10}\lambda^2 + o(\lambda^2),$$

and (4.21) and (4.22) we get that

$$\lim_{n\to\infty} \tilde{\mathbb{E}}_i[\tilde{T}(i)] = \frac{\sqrt{3}}{4\pi}\cdot\frac{16}{10}\varepsilon + o(\varepsilon) = \frac{8}{5}\cdot\frac{\sqrt{3}}{4\pi}\cdot\varepsilon + o(\varepsilon), \tag{4.23}$$

uniformly in the i's with $1 \le i \le M(n)$.

This, combined with (4.18) and the negligibility of $\mathbb{P}_{\mathbb{C}}(B(i))$ (see the line below (4.20)), gives

$$\limsup_{n\to\infty} \max_{1\le i\le M} \mathbb{E}_{\mathbb{C}}[T(i)] \le \frac{8}{5}\cdot\frac{\sqrt{3}}{4\pi}\cdot\varepsilon + o(\varepsilon).$$

By Lemma 4.3.1 this implies Theorem 4.1.1 (b). □